ON ASYMPTOTIC STABILITY OF EQUILIBRIA AND STATIONARY MOTIONS OF MECHANICAL SYSTEMS WITH PARTIAL DISSIPATION

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If it is important that a certain mechanical system be in equilibrium, then it is almost always necessary to choose the system parameters in such a way that the equilibrium be not only stable but asymptotically stable as well. Damping mechanisms are for this reason introduced into certain components of the system, thus assuring the decay of oscillations. It may turn out, however, that for effecting asymptotic stability in a mechanical system it is sufficient to introduce damping in not all but only part of its coordinates. The steady-state motions may also possess such a property. Such systems, asymptotically stable and subject to the action of dissipative forces with incomplete dissipation, are studied in the present paper.

In addition to the well-known theorems on asymptotic stability [1,2], the theorem given in [3] is applied in the paper. A particular but basic case of this theorem on asymptotic stability was formulated in [4]; It was this theorem which accounted for the progress made in studying the problem. We will note that this idea was expressed by Chetaev somewhat earlier and was demonstrated by him with the aid of a particular example [2].

Initially, we recall the theorem of Barbashin and Krasovskii.

Consider the equation of perturbed motion

$$dx_i / dt = X_i (x_1, \ldots, x_n, t) \qquad (0.1)$$

the right-hand sides of which $X_i(x_i, t)$ shall be periodic functions of time t and period θ (or explicitly independent of t) definite and continuous in the region

$$\sup(|x_1|+\ldots+|x_n|)=\|x_s\|\leqslant H; \quad H=\text{const}, \quad \text{or} \quad H=\infty \quad (0.2)$$
$$(-\infty<t<+\infty)$$

In addition, we assume that in each region $\| x_s \| < H_\mu < H$ the functions X_i satisfy the Lipschitz conditions for the variables x_j, i.e.

$$|X_i(x_i'', t)-X_i(x_i', t)|< L_\mu \|x_i''-x_i'\|$$

Theorem. If the perturbed equations are such that it is possible to construct a function $v(x, t)$ (periodic in time t with period θ or explicitly independent of time) which is positive-definite, has an infinitely small upper bound in the region (0.2), satisfies the inequality

$$\text{in the region } \|x_s\|\leqslant H_0, \quad 0\leqslant t<\theta) < \inf(v \text{ for } \|x_s\|=H_1) \; (H_0>H_1>H)$$

and if in addition the function is such that its derivative satisfies the following conditions:

1) Derivative $dv/dt < 0$ in the region (0.2);

2) Derivative dv/dt may be equal to zero only at points of the set M not completely containing the half-trajectories of the system (2.1) $x(x_0, t_0, t)$, $0 < t < \infty$ (with the exception of solution $x_i = 0$), then the solution $x_i = 0$ is asymptotically stable and the region $\| x_s \| \leqslant H_0$ is located in the attraction region for the point $x = 0$.

1. Consider a holonomic mechanical system with stationary constraints subject to forces having a potential function independant of time. Suppose the system is in equilibrium where the combination of second-order terms in the potential function expansion is a quadratic form

$$\delta^2 U=\frac{1}{2}\sum_{i=1}^{n}\beta_{ij}q_iq_j$$

negative-definite with respect to $q_1, \ldots, q_n$, the variations of the holonomic and stationary coordinates of the system in the neighborhood of equilibrium. Such an equilibrium will be stable according to a theorem of Lagrange.

Let the system be subjected to the dissipative forces $R_{n-k+1},\ldots,R_n$ with partial dissipation along the last $n-k$ coordinates $q_{n-k+1},\ldots,q_n$ such that

$$R_{n-k+j}=\frac{\partial}{\partial \dot{q}_j}\frac{1}{2}\sum_{ij=n-k+1}^{n}\theta_{ij}\dot{q}_i\dot{q}_j=\frac{\partial}{\partial \dot{q}_i}\frac{1}{2}F$$

Here F is a negative-definite quadratic form of $\dot{q}_{n-k+1}, \ldots, \dot{q}_n$ with constant coefficients.

If $\delta^2 T$ is a combination of terms of second order in the kinetic energy expansion then, on the strength of (1.1)

$$\frac{d}{dt}(\delta^2 T - \delta^2 U) = F$$

and the motion is stable as well as asymptotically stable with respect to $\dot{q}_{n-k+1}, \ldots, \dot{q}_n$ [5]. According to the theorem of Barbashin and Krasovskii the motion will be asymptotically stable if Equations (1.1) possess no trajectories fully located in the region $q_{n-k+1} = q_{n-k+1}^\circ$, $\ldots, q_n = q_n^\circ$. If the equations of first approximation do not possess such a trajectory, then the motion will be asymptotically stable on the strength of the system of first approximation, which we will take in the form solved with respect to $\ddot{q}_i$

$$\ddot{q}_i = a_{i1}q_1 + \ldots + a_{in}q_n + \sum_{j=n-k+1}^{n} \frac{\partial F}{\partial \dot{q}_j} \beta_{ij}' \qquad (i = 1, 2, \ldots, n) \qquad (1.1)$$

These equations possess the indicated particular solution if in the first $n - k$ equations, where $q_{n-k+1}, \ldots, q_n$ are assumed constant, there exists a particular solution satisfying the equations

$$\ddot{q}_i = a_{i1}q_1 + \ldots + a_{in-k}q_{n-k} + a_{in-k+1}q_{n-k+1}^\circ + \ldots + a_{in}q_n^\circ$$
$$(i = 1, 2, \ldots, n-k)$$
$$0 = a_{i1}q_1 + \ldots + a_{in-k}q_{n-k} + a_{in-k+1}q_{n-k+1}^\circ + \qquad + a_{in}q_n^\circ \qquad (1.2)$$
$$(i = n-k+1, \ldots, n)$$

Any particular solution of the first $n - k$ equations for $q_n = q_n^\circ$, $\ldots, q_{n-k+1} = q_{n-k+1}^\circ, \ldots$ can be expressed as a sum. This particular solution will be formed from the solution of the system

$$\ddot{q}_i = a_{i1}q_1 + \ldots + a_{in-k}q_{n-k}$$

and some constant components $q_1^\circ, \ldots, q_{n-k}^\circ$ which will be found from the equations

$$a_{i1}q_1^\circ + \ldots + a_{in-k}q_{n-k}^\circ = -a_{in-k+1}q_{n-k+1}^\circ - \ldots - a_{in}q_n^\circ \qquad (1.3)$$
$$(i = 1, \ldots, n-k)$$

They will always be found uniquely, since the determinant of the last system is known to be nonzero and $\delta^2 U$ was assumed negative-definite. These constants $q_1^\circ, \ldots, q_{n-k}^\circ$ must necessarily satisfy the system

$$a_{i1}q_1^\circ + \ldots + a_{in}q_n^\circ = 0 \quad (i = n-k+1, \ldots, n) \tag{1.4}$$

and, consequently, must satisfy the system (1.3), (1.4). Therefore $q_i^\circ = 0$ are all zero, since the determinant of the system is nonzero.

Thus the question has been reduced to the existence of a solution for the system

$$\ddot{q}_i = a_{i1}q_1 + \ldots + a_{in-k}q_{n-k} \quad (i = 1, \ldots, n-k)$$

located in the region

$$a_{i1}q_1 + \ldots + a_{in-k}q_{n-k} = 0 \quad (i = n-k+1, \ldots, n)$$

If such a solution exists, then along it one must have

$$= a_{i1}(a_{11}q_1 + \ldots + a_{1n-k}q_1) + \ldots + a_{in-k}(a_{n-k,1}q_1 + \ldots + a_{n-k,n-k}q_{n-k})$$
$$(i = n-k+1, \ldots, n) \tag{1.5}$$

and furthermore all linear forms obtained by twofold, fourfold, ..., $2n$-fold differentiation are equal to zero on the strength of the first $n-k$ equations where $q_{n-k+1} = 0, \ldots, q_n = 0$. These forms are

$$\sum_{i,j,s=1}^{n-k} a_{i1}a_{j1}a_{s1}q_1 + \ldots + a_{in-k}a_{jn-k}a_{sn-k}q_{n-k} = 0 \tag{1.6}$$

$$\sum_{\nu_1,\ldots,\nu_{\mu+2n-k}}^{n-k} a_{\nu_1 1}a_{\nu_2 1}a_{\nu_3 1}, \ldots, a_{\nu_{\mu+2}1}q_1 + \ldots + a_{\nu_1 n-k}, \ldots, a_{\nu_{\mu+2n-k}}q_{n-k} = 0$$

If among them there are $n-k$ independent ones, then this means that such a non-trivial solution does not exist and the motion is asymptotically stable, since on the strength of the first approximation the asymptotic stability can occur only when all characteristic exponents of that system have negative real parts.

Let $q_{n-k+1}, \ldots, q_n$ be expressed by means of normal coordinates $x_1, \ldots, x_n$ as

$$q_i = b_{i1}x_1 + \ldots + b_{in}x_n \qquad (i = n-k+1, \ldots, n) \tag{1.7}$$

The equations of first approximation in normal coordinates are of the form

$$\ddot{x}_i = \lambda_i x_i + \frac{\partial F}{\partial \dot{x}_i}$$

whereby the $dF/\partial\dot{x}_i$ vanish when $\dot{q}_{n-k+1} = \ldots = \dot{q}_n = 0$. Differentiating each of the linear forms (1.7) $2p_i$ times we obtain the equations

$$b_{i1}\lambda_1{}^l x_1 + \ldots + b_{in}\lambda_n{}^l x_n = 0 \qquad (l = 1, \ldots, p_i;\ i = n-k+1, \ldots, n) \quad (1.8)$$

It is easy to note that Equations (1.2), (1.5) in conjunction with $q_{n-k+1} = \ldots = q_n = 0$, (1.6) are equivalent to (1.7) in conjunction with (1.8) since they follow from the same equations differentiated several times in view of the same system of differential equations.

Any minor of order p_i in the ith group of equations (1.7), (1.8) is of the form of a Vandermonde determinant [6]

$$\det\|b_{ij}{}^{pi}\lambda_j{}^{pip_i}\| = \prod_{k<j=j_1,\ldots,jp_1} b_{ij}(\lambda_i - \lambda_k) \qquad (p_i = 0, 1, \ldots, n-1)$$

Consider initially the case $k = 1$. Differentiating $2n - 2$ times the equation

$$b_{n1}x_1 + \ldots + b_{nn}x_n = 0$$

we will obtain n linear forms with the determinant $D = \| b_{nj}\lambda_j{}^{i-1}\|$ $(ij = 1, \ldots, n)$.

According to Vandermonde's theorem

$$D = b_{n1}, \ldots, b_{nn} \prod_{1\leqslant i<j\leqslant n} (\lambda_i - \lambda_j)$$

This determinant will not be zero if there are no zeros among b_{nj} and if there are no equal quantities among $\lambda_1, \ldots, \lambda_n$.

Let $k = 2$. Differentiating $2(p_1 - 1)$ and $2(p_2 - 1)$ times the equations

$$b_{n-1,1}x_1 + \ldots + b_{n-1,n}x_n = 0$$
$$b_{n,1}x_1 + \ldots + b_{nn}x_n = 0$$

we will obtain a system of $p_1 + p_2 = n$ linear forms with the determinant

$$D_2 = \begin{vmatrix} b_{n-1,1} & \ldots & b_{n-1,n} \\ \ldots & \ldots & \ldots \\ b_{n-1,1}\lambda_1{}^{p_1-1} & \ldots & b_{n-1,n}\lambda_n{}^{p_1-1} \\ b_{n1} & \ldots & b_{nn} \\ \ldots & \ldots & \ldots \\ b_{n1}\lambda_1{}^{p_2-1} & \ldots & b_{nn}\lambda_n{}^{p_2-1} \end{vmatrix}$$

Taking the minors of order p_1 from the first p_1 rows, multiplying them by the minors of order p_2 from the last p_2 rows and adding we will obtain, according to Laplace's theorem, the required determinant.

If the minor of order p_1 consists of the columns with numbers j_1^s, ..., $j_{p_1}^s$, then the minor of order p_2, by which it is multiplied, consists of the columns with numbers i_1^s, ..., $i_{p_2}^s$ where the numbers j_1^s, ..., $j_{p_1}^s i_1^s$, ..., $i_{p_2}^s$ assume values of 1 to n. Their product can be expressed in the form

$$b_{n-1,j_1^s}, \ldots, b_{n-1,j_{p_1}^s} b_{ni_1^s}, \ldots, b_{ni_{p_2}^s} \prod_{\substack{i<j \\ j_1^s,\ldots,j_{p_1}^s}} (\lambda_i - \lambda_j) \prod_{\substack{i<j \\ i_1^s,\ldots,i_{p_2}^s}} (\lambda_i - \lambda_j)$$

Summing these expressions over all possible s we obtain the determinant D_2. If among the principal frequencies of the system there are three equal to each other, then the last determinant is equal to zero. Indeed, in any division of λ_1, ..., λ_n into two groups, the equality of any two numbers in any one group results in vanishing of the above products. It is easy to show that if it is impossible to indicate more than two equal principal frequencies λ_i, then by a suitable choice of two linear combinations $q_{n-1} = q_n = 0$ it is possible to succeed in preventing their determinant D_2 from vanishing.

Indeed, let us consider pairs of equal principal frequencies and divide the frequencies into two groups in such a way that no pair would completely fall into any one group. If there are q such pairs, then denoting by λ_1, ..., λ_q the principal frequencies of the pairs we will consider two linear forms

$$b_{n-1,1}x_1 + \ldots, + b_{n-1,q}x_q = 0$$
$$b_{n,q+1}x_{q+1} + \ldots b_{nn}x_n = 0$$

Differentiating the first form $2(q-1)$ times, and the second $2(n-q-1)$ times, we will obtain a system of forms with the determinant

$$D_2 = b_{n-1,1}, \ldots, b_{n-1,q} b_{nq+1}, \ldots, b_{nn} \prod_{1\leqslant i<j\leqslant q} (\lambda_i - \lambda_j) \prod_{q+1\leqslant i<j\leqslant n} (\lambda_i - \lambda_j)$$

It will not be equal to zero if no $b_{n-1,i}$, b_{ni} is equal to zero.

If we have k linear forms

$$b_{i1}x_1 + \ldots + b_{in}x_n = 0 \qquad (i = n-k+1, \ldots n)$$

while among $\lambda_1, \ldots, \lambda_n$ there are $k+1$ equal to each other, then for any $p_1 + \ldots + p_k = n$ all linear forms obtained by means of $2(p_j - 1)$-fold differentiation from the jth form in conjunction with the given k_i forms will be linearly dependent. Indeed, the determinant of these forms can be represented, in accordance with Laplace's theorem, in the form of a sum of products of the type

$$\beta_\nu \prod_{1 \leqslant i < j \leqslant p_1} (\lambda_i - \lambda_j), \ldots, \prod_{p_{k-1} < i < j \leqslant p_k} (\lambda_i - \lambda_j)$$

If among λ_i there are $k+1$ equal to each other, then for any distribution of them into k groups, one group will certainly contain a pair of equal λ_i. Therefore, the products will vanish, including the determinant D_k. If there are no more than k equalities, then, distributing all λ into k groups

$$\lambda_1, \ldots, \lambda_{p_1}, \quad \lambda_{p_{k+1}}, \ldots, \lambda_{p_2}, \ldots, \lambda_{p_{k-1}+1}, \ldots, \lambda_n$$

in such a way that no one group would contain two equal numbers, we obtain the result that the linear system

$$\begin{gathered} b_{n-k+1,1}x_1 + \ldots + b_{n-k+1,p_1}x_{p_1} = 0 \\ b_{n-k+2,p_1+1}x_{p_1+1} + \ldots + b_{n-k+2,p_2}x_{p_2} = 0 \\ b_{n,(p_{k-1}+1)},x_{p_{k-1}+1} + \ldots + b_{n,n}x_n = 0 \end{gathered}$$

which is obtained after $2(p_j - 1)$-fold differentiation of the $(n-k+j)$-th form, will lead to a system of forms with the determinant

$$D_k = b_{n-k+1,1}, \ldots, b_{n-k+1,p_1}, \ldots, b_{nn} \prod_{1 \leqslant i < j \leqslant p_1} (\lambda_i - \lambda_j), \ldots, \prod_{p_{k-1} < i < j \leqslant n} (\lambda_i - \lambda_j)$$

which is nonzero if no b_{ij} is nonzero.

As is known, any normal oscillation of a system of first approximation can be expressed in the form

$$x_i = A_i \cos \sqrt{|\lambda_i|}\, t + B_i \sin \sqrt{|\lambda_i|}\, t$$

In order that this solution at all times satisfy the equality

$$b_{i1}x_1 + \ldots + b_{in}x_n = 0 \qquad (i = n-k+1, \ldots, n)$$

it is necessary that these equalities be at all times satisfied by all of their time derivatives as well.

Differentiating each of the equalities $2(p_i - 1)$ times and letting

$t = 0$, we find that A_i, B_i are dependent on the equalities

$$b_{i1}\lambda_2^{s-1}A + \ldots + b_{in}\lambda_n^{s-1}A_n = 0 \qquad (i = n - k + 1, \ldots, n)$$
$$b_{i1}\lambda_1^{s-1}B_1 + \ldots + b_{in}\lambda_n^{s-1/2}B_n = 0 \qquad (s = 1, \ldots, p_i)$$

Multiplying each jth column of the determinant in the second system by $\lambda_j^{1/2}$, it is easy to see that we will obtain identically the determinant of the first system. It is also easy to see that it coincides with the determinant which was studied above. If this determinant is nonzero, then it is clear that no non-trivial solution satisfying these equalities exists.

All further differentiations lead to the conclusion that A_i, B_i, $\lambda_i^{1/2}$ satisfy the one and the same infinite system of equations with the matrix

$$(b_{ij}\lambda_j^{s-1}) \qquad (i = n - k + 1, \ldots, n;\ j = 1, \ldots, n;\ s = 1, 2, \ldots)$$

which represents identically the matrix of the system (1.7), (1.8); the system (1.7), (1.8) can be obtained by linear substitution from Equations (1.2), (1.5), (1.6) and the equations $q_{n-k+1} = q_n = 0$. It is obvious that these equations will certainly have a non-trivial solution if among the principal frequencies of the system there are $k + 1$ equal to $\lambda_1 = \ldots = \lambda_{k+1}$, and if it is assumed that $A_{k+2} = \ldots = A_n = 0$, while for the definition of $A_1, \ldots, A_{k+1}$ one writes the equation

$$b_{i1}A_1 + \ldots + b_{ik+1}A_{k+1} = 0$$

These equations will necessarily have a non-trivial solution, and in the fulfilment of these equations and the condition $B_i = A_i/\lambda_i^{1/2}$ the linear forms $q_{n-k+1}, \ldots, q_n$ will become zero identically in t, as also their time derivatives. Consequently, in the presence of $k + 1$ equalities of λ_i in the system (1.1), there will always be at least one non-vanishing solution.

Let us formulate the result.

Theorem. 1) In order that partial dissipation along $k < n$ coordinates of a system render asymptotically stable in the first approximation the isolated and stable position of equilibrium of a mechanical system, in the presence of only potential forces, the potential function expansion of which in the neighborhood of the equilibrium begins as a definite negative quadratic form, it is necessary and sufficient that the equations $q_{n-k+1} = \ldots = q_n = 0$ and Equations (1.8) which are obtained from them by differentiation on the strength of the system of first approximation, would have a matrix of rank n. This condition will be sufficient for asymptotic stability on the strength of the exact equations.

2) If there are $k + 1$ equal frequencies among the principal frequencies of first approximation, then no dissipation along any k coordinates will make the equilibrium asymptotically stable in the first approximation.

3) If there are no more than k equal principal frequencies, then it is always possible to indicate k such generalized coordinates, the introduction of partial dissipation along which will result in asymptotic stability of the equilibrium $q_i = 0$.

The theorem of Barbashin and Krasovskii can be interpreted as follows when it is applied to a mechanical system of the type considered.

If one imposes upon a mechanical system certain constraints

$$b_{i1}x_1 + \ldots + b_{in}x_n = 0$$

where $x_1, \ldots, x_n$ are normal coordinates of the system, then the Lagrange equations of the first kind will be of the form

$$\ddot{x}_i = \lambda_i x_i + \sum_{j=n-k+1}^{n} \theta_j b_{ji} + X_i \tag{1.9}$$

Where X_i are terms of higher than first order.

In accordance with the proof of Barbashin and Krasovskii it is sufficient for asymptotic stability that there be no solution of the system

$$\ddot{x}_i = \lambda_i x_i + X_i \qquad (i = 1, \ldots, n)$$

fully located in the region

$$b_{i1}x_1 + \ldots + b_{in}x_n = 0 \qquad (i = n - k + 1, \ldots, n)$$

if the dissipation is introduced along the coordinates

$$q_i = b_{i1}x_1 + \ldots + b_{in}x_n$$

Let these conditions be satisfied. This means that the system (1.9) has no solution along which

$$\sum_{j=n-k+1}^{n} \theta_j b_{ji} = 0 \qquad (i = 1, \ldots, n)$$

at all times during motion. It follows necessarily from these equations that there is no solution along which all θ_j would become zero. The quantities θ_j may be interpreted as the constraint reactions, as follows from the theorem.

If it is possible to apply additional constraints upon the type of system considered in such a way that there should exist no motion in the constrained system along which all reactions of new constraints would vanish, then the introduction of dissipative forces in the original system acting along those coordinates which have been removed in the constrained system, by the application of the additional constraints, will make the equilibrium of the mechanical system asymptotically stable.

2. In order to limit ourselves to the most easily proved consequences of the theorem of Barbashin and Krasovskii let us consider the system (0.1) in the form

$$\frac{dx_i}{dt} = p_{i1}x_1 + \ldots + p_{in}x_n + X_i \tag{2.1}$$

Here p_{sj} are bounded periodic functions of time, while X_i are holomorphic, with respect to x_i, functions with periodic coefficients, continuous and bounded.

Let dv/dt become zero only under the conditions

$$F_1(x_1, \ldots, x_n, t) = \ldots = F_k(x_1, \ldots, x_n t) = 0$$

Naturally, if there exists a half-trajectory $x_i(t)$ satisfying these equalities, then along it the conditions

$$\frac{d^i F_s}{dt^i} = 0 \qquad (s = 1, \ldots, k;\ i = 1, 2, \ldots)$$

are necessarily satisfied, and consequently there exists a non-trivial solution of the equation

$$\Psi = \sum_{i,s} \left(\frac{d^i F_s}{dt_i} \right)^2 = 0 \qquad (i = 0, 1, \ldots;\ s = 1, \ldots k)$$

If each of the derivatives is now thought of as taken on the strength of Equations (2.1), then the desired solution must satisfy an infinite system of equations with variables $x_1, \ldots, x_n, t$. Such a case will certainly not be found if the function Ψ for any fixed $t > 0$ is of definite sign with respect to $x_1, \ldots, x_n$.

On the basis of the known theorem of Liapunov the system (2.1) can, by means of a non-special linear transformation, be transformed into

$$\frac{dy_i}{dt} = \lambda_i y_i + Y_i$$

Here λ_i are complex numbers with non-positive real parts which split into pairs of complex-conjugates.

Let $F_1, \ldots, F_k$ after substitution become $\Phi_1, \ldots, \Phi_k$, where their expansions in $y_1, \ldots, y_n$ are of the form

$$\Phi_i = b_{i1} y_1 + \ldots + b_{in} y_n + U_i$$

and where $b_{ip} + b_{i,p+1}$ are real numbers, $b_{ip} - b_{i,p+1}$ are purely imaginary.

If the system of equations

$$b_{i1} \lambda_1^s y_1 + \ldots + b_{in} \lambda_n^s y_n = 0 \qquad (s = 0, 1, \ldots;\ i = 1, \ldots, k)$$

has a matrix of rank n, then the system of first approximation, as well as the original system (2.1), will be asymptotically stable. If it is of lower rank than n, then the system of first approximation will not be asymptotically stable. This will always occur if among $\lambda_1, \ldots, \lambda_n$ there are $k + 1$ equal to each other. If such a case does not occur, then it is always possible to indicate such linear parts of the function Φ_i that asymptotic stability would take place for all variables.

If it should happen that the rank of the indicated matrix is equal to $n - m$, then choosing $n - m$ independent equations from the system $d^b\Phi/dt^b = 0$ and solving them with respect to the first $n - s$ variables, we substitute the result into the remaining equations. The expansions of the result of the substitution will begin with the quadratic forms of m variables. If among the quadratic forms one can choose a linear combination with coefficients which are dependent on time in such a way that it represents a positive-definite form of its variables, then surely the asymptotic stability will take place.

3. Consider a mechanical system, the kinetic energy T of which does not depend explicitly on t, $q_{n-k+1}, \ldots, q_n$, the last generalized coordinates, while the potential energy is of the form

$$U = U_1(q_1, \ldots, q_{n-k}) + F_{n-k+1} q_{n-k+1} + \ldots + F_n q_n$$

Here $F_{n-k+1}, \ldots, F_n$ are constants. Let the system be subject also to dissipative forces with the dissipative function

$$F = \frac{1}{2} \sum_{ij=n-l+1}^{n} \beta_{ij} \dot{q}_i \dot{q}_j \qquad (l \geqslant k)$$

embracing the last k coordinates and a few others. If is shown in [7] that the given system has a stationary solution

$$q_1 = \ldots q_{n-k} = 0, \quad q_{n-k+1} = \dot{q}^{\circ}_{n-k+1}(t-t_0), \ldots, q_n = \dot{q}^{\circ}_n (t-t_0)$$

where $\dot{q}_{n-k+1}{}^{\circ}, \ldots, \dot{q}_n{}^{\circ}$ are certain constants if they satisfy the equations

$$\frac{\partial U_1}{\partial q_i} = 0 \qquad (i = 1, 2, \ldots, n-k), \qquad F_i + \sum_{j=n-k} \beta_{ij}\, \dot{q}_i{}^{\circ} = 0 \quad (i = n-k+1, \ldots, n)$$

It is shown also that if the quadratic part of the $T - U$ expansion in the neighborhood of the stationary motion is a positive-definite function with respect to the variations of the coordinates and velocities, then for complete dissipation the stationary motion will be asymptotically stable.

By means of a method analogous to that of Section 1, one can show that the asymptotic stability in the presence of partial dissipation along the last coordinates will not exist only in the case where the equations of perturbed motion will possess trajectories located in the region $q_{n-k+1} = \ldots = q_n = 0$.

The test for sign-definiteness of the quadratic part of the $T - U$ function expansion can also be simplified. As was shown by Routh [8], the function T expressed by means of the variables $q_1, \ldots, q_{n-k}$, $p_{n-l+1}, \ldots, p_n$

$$2T = \sum_{ij=1}^{n-k} \alpha_{ij}\, \dot{q}_i \dot{q}_j + \sum_{ij=n-k+1}^{n} \gamma_{ij}\, p_i p_j$$

will not contain terms with products of velocity by the impulse.

Since $\dot{q}_1{}^{\circ} = \ldots \dot{q}_{n-k}{}^{\circ} = 0$, it is easy to note that

$$\delta^2 T - \delta^2 U = \frac{1}{2} \sum_{ij=1}^{n-k} \alpha_{ij}{}^{\circ}\, \dot{q}_i \dot{q}_j + \frac{1}{2} \sum_{ij=n-k+1}^{n} \gamma_{ij}{}^{\circ}\, \delta p_i\, \delta p_j +$$

$$+ \sum_{\substack{ij=n-k+1 \\ s=1,\ldots,n-k}} p_i{}^{0} \left(\frac{\partial \gamma_{ij}}{\partial q_s}\right)^{\circ} \delta p_j\, q_s + \delta^2 \left[\frac{1}{2} \sum_{ij=n-k+1}^{n} \gamma_{ij}\, p_i{}^{\circ} p_j{}^{\circ} - U_1\right]$$

The first sum is a positive-definite quadratic form of the velocities, the second is a variation of impulses, while the third and fourth sums contain no cyclical velocities. Therefore, the conditions of sign-definiteness are reduced to the positiveness of $\Delta_{k+1}, \ldots, \Delta_n$ diagonal minors of the quadratic form discriminant consisting of the last three sums. These minors border the discriminant of the second sum, known to be positive.

As already shown above, the motion will be asymptotically stable with respect to the last l coordinates, where $l \geqslant k$, since the existence of the nonperturbed motion of the type indicated presupposes the presence of dissipation along the last k coordinates.

In the case when the nonperturbed motion is not asymptotically stable, the equations of motion must necessarily have a particular solution

$$q_i = q_i(t) \qquad (i \leqslant n - k), \qquad q_i = \dot{q}_i^\circ (t - t_0) \qquad (i > n - k)$$

whereupon the first $n - k$ functions are not all zero.

The equations which must be satisfied by these $n - k$ functions can be obtained from the Lagrange function of the type

$$L = T(\dot{q}_1, \ldots, \dot{q}_{n-k}, \dot{q}^\circ_{n-k+1}, \ldots, \dot{q}_n^\circ, q_1, \ldots, q_{n-k}) + U_1$$

and the last k equations will be of the form

$$\frac{d}{dt}\frac{dT}{d\dot{q}_i^\circ} = 0 \qquad (i = n - k + 1, \ldots, n)$$

or

$$\sum_{i=1}^{n-k} \alpha_{ij} \dot{q}_i + \sum_{i=n-k+1}^{n} \alpha_{ij} \dot{q}_i^\circ = c_j \qquad (j = n - k + 1, \ldots, n)$$

while the first equations

$$\frac{d}{dt}\Big(\sum_{j=1}^{n-k} \alpha_{ij} \dot{q}_j + \sum_{j=n-k+1}^{n} \alpha_{ij} \dot{q}_i^\circ\Big) - \frac{\partial T}{\partial q_i} = \frac{\partial U_1}{\partial q_i} \qquad (i = 1, \ldots, n - k)$$

Finally, we are led to the problem, whether the system with the kinetic energy of the form

$$T' = T_2 + T_1 = \sum_{ij=1}^{n-k} \alpha_{ij} \dot{q}_i \dot{q}_j + \sum_{\substack{i \leqslant n-k \\ j \geqslant n-k}} \alpha_{ij} q_i q_j^\circ$$

and the potential function of the form

$$U' = U_1 - \sum_{ij=n-k+1}^{n} \alpha_{ij} \dot{q}_i^\circ \dot{q}_j^\circ$$

possess a motion for which the expressions

$$\sum_{ij_2=1}^{n-k} \alpha_{ij} q_j + \sum_{j=n-k+1}^{n} \alpha_{ij} q_i^\circ = c_i \qquad (i = n - k + 1, \ldots, n) \qquad (3.1)$$

would remain constant and possibly $q_{n-l+1} = \dots q_{n-k} = 0$, where this motion would at all times remain arbitrarily close to the origin of the coordinates. If the quadratic forms

$$\delta^2 T_2 = \sum_{i=1}^{n-k} \dot{x}_i^2, \qquad \delta^2 U' = \sum_{i=1}^{n-k} \lambda_i x_i^2$$

are put into a canonical form by a linear transformation which is general to both forms, and if after this transformation the expression for T will become

$$T_1' = \sum_{\substack{i \leqslant n-k \\ j > n-k}} \beta_{ij} q_j^{\circ} \dot{x}_i$$

then the equations of first approximation will be of the form

$$\ddot{x}_i + \sum_{\substack{i,\, s \leqslant n-k \\ j > n-k}} \left(\frac{\partial \beta_{ij}}{\partial x_s}\right)^{\circ} q_i^{\circ} \dot{x}_s = \lambda_i x_i \tag{3.2}$$

while the variations of their particular integrals (3.1) will be of the form

$$\sum_{i=1}^{n-k} \beta_{i\beta} \dot{x}_i + \sum_{i,\, s \leqslant n-k} \left(\frac{\partial \beta_{ij}}{\partial x_s}\right)^{\circ} \dot{q}_i^{\circ} x_s = c_j \qquad (j = n-k+1, \dots, n) \tag{3.3}$$

If in the last equations one can define $l \leqslant k$ variables x_i, $\dot{x}_i$ as functions of the remaining ones and $c_{n-k+1}, \dots, c_n$, then after substitution of these solutions into (3.2) we will obtain a nonhomogeneous system. Since a homogeneous system has no first time-independent linear integrals, the desired particular solution will consists of a sum of a particular solution of a homogeneous system plus a certain constant vector. Therefore, if such a solution exists, then there exists necessarily a particular solution of the homogeneous system and vice versa.

Thus, the question is reduced to the following: is there a solution for the system (3.2) restricted by the conditions (3.3) for $c_{n-k+1} = \dots = c_n = 0$ and also $q_{n-l+1} = \dots q_{n-k} = 0$?

It is known not to exist if the rank of the matrix from these forms, obtained by infinite differentiation on the strength of Equation (3.2), is equal to $n - l$.

Thus, if the rank of the system matrix, obtained from Equations (3.3) $q_{n-l+1} = q_{n-k} = 0$ by infinite differentiation on the strength of

Equations (3.2), is $n - l$, then the motion will be asymptotically stable and only then will the motion be asymptotically stable in the first approximation.

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